Chapter 5

Elementary probability

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Introduction:

- Probability theory is the foundation upon which the logic of inference is built.
- It helps us to cope up with uncertainty.
- In general, probability is the chance of an outcome of an experiment. It is the measure of how likely an outcome is to occur.
- In Statistics, sensible numerical statements can be made about uncertainty / certainty and apply different approaches to calculate these uncertainties/certainties.
- These numerical measures of uncertainties/certainties are called probabilities of occurrence of the events.

Key terms

Key Terms in Probability

- **Experiment:** Any process of observation or measurement or any process which generates well defined outcome.
- **Probability Experiment (Random Experiment):** It is an experiment that can be repeated any number of times under similar conditions
- **Outcome:** The result of a single trial of a random experiment
- **Sample space, S:** a collection/set of all possible outcomes of an experiment
- **Event, E:** Any subset of sample space, S.
- **Simple /elementary event:** An event consisting a single out come.
- **Equally Likely Events:** Events which have the same chance of occurring.

Key terms

Key terms in probability

- **Certain event:** An event which is sure to occur.
- **Impossible event:** An event which can't occur.
- **Complement of an event:** The complement of event A (denoted by $A' or A^c$), *consists of all the sample points inthe sample space that not in*
- **Independent events:** Two events are said to be independent if the occurrence of one is not affected by, and does not affect, the other. If two events are not independent, then they are said to be **dependent**.
- **Equally likely out comes:** If each out come in an experiment has the same chance to occur, then the outcomes are said to be equally likely

Counting Rule

In order to calculate probabilities, we have to know

- The number of elements of an event
- The number of elements of the sample space
- In order to determine the number of outcomes, one can use several rules of counting.
  - The addition rule
  - The multiplication rule
  - Permutation rule
  - Combination rule

Addition Rule

If event **A** can occur in m possible ways and event **B** can occur in **n** possible ways, there are **m+n** possible ways for either event **A or event B to occur**, but only if there are **—no events in common between them**.

$$n(AorB) = nA + nB - n(A \cap B) \tag{1}$$

Example: A student goes to the nearest snack to have a breakfast. He can take tea, coffee, or milk with bread, cake and sandwich. How many possibilities does he have?

The Multiplication Rule

The Multiplication Rule

- If a choice consists of k steps of which the first can be made in n1 ways, the second can be made in n2 ways, the $K^{th} canbemadeinn_k ways, thenthewholechoicecanbemadeinn_1 \times n_2 \times n_3 .... n_x ways.$
- **Example :**A student has two shoes, three trousers and three jackets. In how many can be dressed?
- **Example 2:** The digits 0, 1, 2, 3, and 4 are to be used in 4 digit identification card. How many different cards are possible if:
  1. Repetitions are permitted.
  2. Repetitions are not permitted.

Permutations

Permutation

The number of possible permutations is the number of different orders in which particular events occur. The number of possible permutations are:

$$N_{pr} = \frac{n!}{(n-r)!} \tag{2}$$

where **r** is the number of events in the series,
n is the number of possible events, and
n! denotes the factorial of n = the product of all the positive integers from 1 to n.

Combinations

Combinations

- When the order in which the events occurred is of no interest, we are dealing with combinations. The number of possible combinations is

$$N_c = \binom{n}{k} = \frac{n!}{r!(n-r)!} \tag{3}$$

- where **r** is the number of events in the series,
  n is the number of possible events, and
  n! denotes the factorial of n = the product of all the positive integers from 1 to n.

Approaches to measuring Probability

There are four different conceptual approaches to study probability theory.

- **The classical approach.**
- **The frequencies approach.**
- **The axiomatic approach.**
- **The subjective approach.**

The Classical Approach

- if there are m outcomes in a sample space (universal set), and all are equally likely of being the result of an experimental measurement, then the probability of observing an event (a subset) that contains s outcomes is given by $s\frac{}{m}$
- All outcomes are equally likely and mutually exclusive.
- Total number of outcome is finite, say N.
- The probability that event A occur denoted P(A) is defined as:

$$p(A) = \frac{N_a}{N} = \frac{n(A)}{n(s)} = \frac{No.\ ofOutcome\ favourable\ for\ A}{Total\ number\ of\ outcomes} \tag{4}$$

The Classical Approach

Limitation

- If it is not possible to enumerate all the possible outcomes for an experiment.
- If the sample points (outcomes) are not mutually independent.
- If the total number of outcomes is infinite.
- If each and every outcomes is not equally likely.

Example

Example 5: A fair die is tossed once. What is the probability of getting

i Number 4?

ii An odd number?

iii Number greater than 4?

iv Either 1 or 2 or . Or 6

v Number 8 ?

The Frequencies Approach

- **Relative frequency probability:** If some process is repeated a large number of n times, and some resulting event E occurs m times, the relative frequency of E (m/n) will be approximately equal to the probability of E
- Symbolically:

$$Pr(E) = \frac{m}{n}$$

  - E.g. Suppose that of 158 people who attended a dinner party, 99 were ill due to food poisoning. The probability of illness for a person selected at random is

$$Pr(illness) = \frac{99}{158} = 0.63$$

$$or \quad = 63\%$$

Axiomatic Approach

Let E be a random experiment and S be a sample space associated with E. With each event A

i $P(A) \geq 0$

ii $0 \leq p(A) \leq 1$

ii $P(s)=1$

v If A and B are mutually exclusive events, $P(A \cup B) = P(A) + P(B)$
$P(A') = 1 - P(A)$

v $P(\phi) = 0$, ϕ *is impossible event*

Subjective Approach

- Subjective probability: measures the confidence or a wish that a particular individual has in the truth of a particular proposition
- **E.g.** If some one says that he is 95 % certain that a cure for AIDS will be discovered within 5 years, then he means that Pr(discovery of cure of AIDS within 5 years) = 95%
- It is usually set from intuition, educated guesses, or estimates
- Although the subjective view of probability has enjoyed increased attention over the years, it has not been fully accepted by scientists

- **Conditional Events:** If the occurrence of one event has an effect on the next occurrence of the other event then the two events are conditional or dependant events.

- The formula for calculating a sample conditional probability is easy to use:

$$p(Eve/cond\ event) = \frac{No.of\ \ observation\ \ for\ \ both\ \ event}{No.of\ \ observations\ \ for\ \ whichcond.event\ \ occurs}$$

$$p(A/B) = \frac{P(A \cap B)}{P(B)} \tag{5}$$

- **Example:** For the retinopathy data, the conditional probability of retinopathy, given exposure to light, is:

Independent events

- Two events, A and B, are independent if the occurrence or non-occurrence of either of one does not affect the probability of the occurrence of the other.
- Two events A and B are independent if and only if

$$P(A \cap B) = P(A) \times P(B) \tag{6}$$

Example: Given that P (A) = 0.4, P (B) = 0.2,
Solution:

- Are A and B independent?

$$\begin{aligned} P(A \cap B) = P(A) \times P(B) \\ = 0.4 \times 0.2 \\ = 0.08 \end{aligned}$$

Hence, A and B are independent

Example:2 P(C) = 0.5, P (D) = 0.3, P(C∩D) = 0.10.

- Are C and D independent?

Solution:

$$
\begin{aligned}
P(C \cap D) &= P(A) \times P(D) \\
&= 0.5 \times 0.3 \\
&= 0.15 \\
& \quad \therefore P(C \cap D) \neq 0.10
\end{aligned}
$$

Hence, C and D are dependent

Some probability axioms, postulates and theorems

- the probability of an event is a non-negative real number P(event)≥ 0.
- P(S)=1
- for any two mutual exclusive event $\mathrm{P}(\mathrm{A}\cup B) = p(A) + P(B)$
- If $\mathrm{A}_1, A_2, , A_k are\ pair\ wise\ mutually\ exclusive\ events\ in\ S,\ then:$

$$P(A_1 \cup A_2 \cup A_3 ... \cup A_k) = P(A_1) + P(A_2) + P(A_3) + ...P(A_k)$$

- If A is an event in a sample space S, and is its complement, then $\mathrm{P}(\mathrm{A}^{'}) = 1 - P(A)$

- For any sample space, S , P(ϕ) $= 0$
- An event which cannot occur (or an impossible event) has probability of 0
- for any event probability of even A must be $0 \leq P(A) \leq 1$

Example: The probabilities that a student will score an A, B, C, D or F in a given course are 0.22, 0.32, 0.25, 0.15 and 0.06 respectively. Find the probability that the student will get:

- a grade lower than C
- a pass grade

. **Solution:**
Since the five events are mutually exclusive,

- P (failing) = P (D or F) = P (D) + P (F) = 0.15 + 0.06= 0.21
- P (passing) = P (A or B or C) = P (A) + P (B) + P(C) = 0.22 + 0.32 + 0.25= 0.79

Example: The probability that a person gets affected by disease X is 0.08, that a person gets affected by disease Y is 0.05, and that a person gets affected by both diseases is 0.02, in a given community. Then, Find the probability that a randomly selected person from this community:

i Gets affected by either or both diseases;

ii Gets affected by only one of the diseases.

Solution: Let A be the event that a person gets affected by disease X, and B be the event that a person gets affected by disease Y We are given that P (A) = 0.08, P (B) = 0.05 and P(A$\cap B$) = 0.02

$$P(getting\ either\ or\ both) = P(A \cup B)$$
$$= P(A) + P(B) - P(A \cap B) = 0.08 + 0.05 - 0.02 = 0.011$$

$$P(getting\ only\ one\ of\ the\ diseases) = P(A \cap \bar{B}) \cap (B \cap \bar{A})$$
$$= P(A \cap \bar{B}) + P(B \cap \bar{A})$$
$$BUT = p(only\ X) = P(A \cap \bar{B}) = P(A) - P(A \cap B) = 0.06$$
$$= p(only\ y) = P(B \cap \bar{A}) = P(B) - P(A \cap B) = 0.03$$
$$\therefore P(getting\ only\ one\ of\ the\ diseases) = 0.06 + 0.03 = 0.09$$

Unconditional Probability

- When the size of the total group or grand total (n) serves as the denominator to calculate a probability, the probability is termed as unconditional probability.

- **Example:** A study was conducted to investigate the effect of prolonged exposure to bright light on retinal damage in premature infants. Eighteen out of 21 premature infants exposed to bright light developed retinopathy, while 21 of 39 premature infants exposed to reduced light level developed retinopathy. What is the probability of developing retinopathy?

solution:

$$p(retinopaty) = \frac{No.\ of\ \ infant\ \ with\ \ retinopaty}{Total\ \ number\ \ of\ \ infant} \tag{7}$$

$$= \frac{18+21}{21+39} \tag{8}$$

$$= 0.65 \tag{9}$$

$$\tag{10}$$

unconditional conditional probabilities 3

- Are probabilities that are based on the knowledge that some other event has occurred. In this case the subset of the total group is taken as a denominator
- We want to compare the probability of retinopathy, given that the infant was exposed to bright light, with that the infant was exposed to reduced light
- Exposure to bright-light and exposure to reduced-light are conditioning events (i.e. events we want to take into account when calculating conditional probabilities)
- Conditional probabilities are denoted by P (A/B) (read as Probability of A given B) or P (Event/Conditioning event).

- The formula for calculating a sample conditional probability is easy to use :

$$p(Eve/cond\ event) = \frac{No.of\ \ observation\ \ for\ \ both\ \ event}{No.of\ \ observations\ \ for\ \ whichcond.event\ \ occurs}$$

$$p(A/B) = \frac{P(A \cap B)}{P(B)}, if\ \ P(B) > 0 \tag{11}$$

Example: For the retinopathy data, the conditional probability of retinopathy, given exposure to light, is:

$$p(retino/expo\ Bright.light) = \frac{No.of\ \ infants\ with\ both\ event}{No.of\ \ infants\ exposure\ to\ Bright\ light} = \frac{18}{21} = 0.86$$

$$p(retino/expos\ reduce.light) = \frac{No.of\ \ infants\ with\ both\ event}{No.of\ \ infants\ exposure\ to\ reduce\ light} = \frac{21}{39} = 0.54$$

The conditional probabilities suggest that premature infants exposed to bright light have a higher risk of retinopathy than premature infants exposed to reduced light.

Multiplication Rule of probability 1

- The definition of P (A/B) can also be used to derive general multiplication rule.
- If A and B are any two events in a sample space S, then

$$P(A \cap B) = P(A) \times P(A/B), P(B) \neq 0$$
$$P(A \cap B) = P(B) \times P(B/A), P(A) \neq 0$$

- If A, B and C are any three events in a sample space S, such that P (A) 0 and

$$P(A \cap B) \neq 0, P(A \cap B \cap C)$$
$$= P(A) \times P(B/A) \times P(c/A \cap B)$$

Example:

Table 1 shows the frequency of cocaine use by gender among adult cocaine users

Life time frequency	Male	Female	Total of cocaine use
1-19 times	32	7	39
20-99 times	18	20	38
more than 100 times	25	9	34
Total	75	36	111

Questions:

1. What is the probability of a person randomly picked is a male?
2. What is the probability of a person randomly picked uses cocaine more than 100 times?
3. Given that the selected person is male, what is the probability of a person randomly picked uses cocaine more than 100 times?
4. Given that the person has used cocaine less than 100 times, what is the probability of being female?
5. What is the probability of a person randomly picked is a male and uses cocaine more than 100 times? **Answer:**
   - Pr(m)=Total adult males/Total adult cocaine users =75/111 =0.68 .
   - Pr(c¿100)=All adult cocaine users more than 100 times/ Total adult cocaine users=34/111=0.31.
   - Pr (c¿100)=25/75=0.33.
   - Pr(f¡100)=(7+20)/36=27/36=0.75.
   - Pr(m n c¿100)= Pr(m) Pr (c¿100)=75/111 25/75=25/111=0.23.